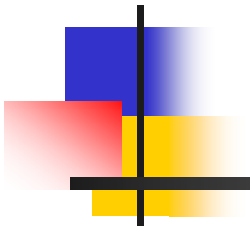
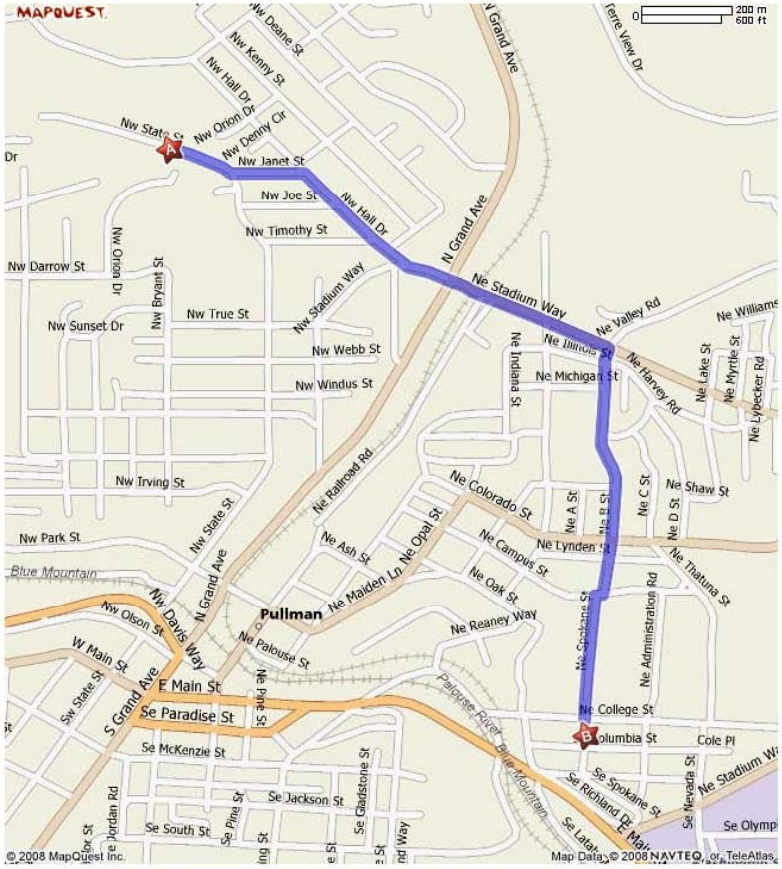


The Shortest Path Problem

Shortest-Path Algorithms

- Find the “shortest” path from point A to point B
- “Shortest” in time, distance, cost, ...
- Numerous applications
  - Map navigation
  - Flight itineraries
  - Circuit wiring
  - Network routing



Shortest Path Problems

Weighted graphs:

- Input is a weighted graph where each edge (v_i,v_j) has cost $c_{i,j}$ to traverse the edge
- Cost of a path $v_1v_2...v_N$ is $\sum_{i=1}^{N-1} c_{i,i+1}$
- Goal: to find a smallest cost path

Unweighted graphs:

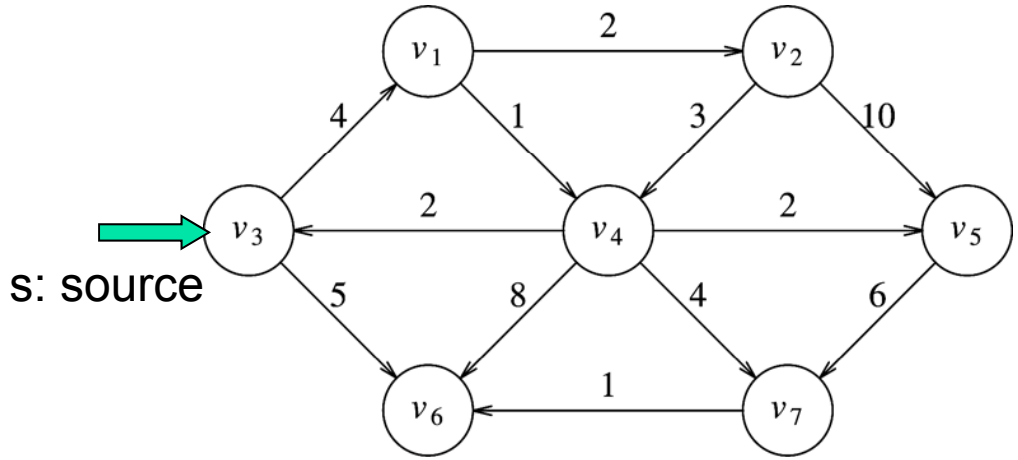
- Input is an unweighted graph
  - ⇒ i.e., all edges are of equal weight

  Goal: to find a path with smallest number of hops

Shortest Path Problems

Single-source shortest path problem

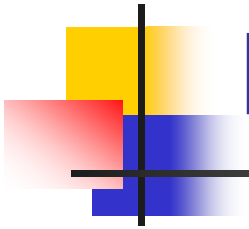
- Given a weighted graph G=(V,E), and a source vertex s, find the minimum weighted path from *s* *to every other vertex* in G



Some algorithms:

Weighted:
Dijkstra's algo

Unweighted:
Simple BFS



Point to Point SP problem

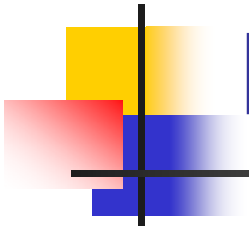
Given G(V,E) and two vertices A and B, find a shortest path from A (source) to B (destination).

Solution:

1) Run the code for Single Source Shortest Path using source as A.

2) Stop algorithm when B is reached.

All Pairs Shortest Path Problem



Given G(V,E), find a shortest path between all pairs of vertices.

Solutions:

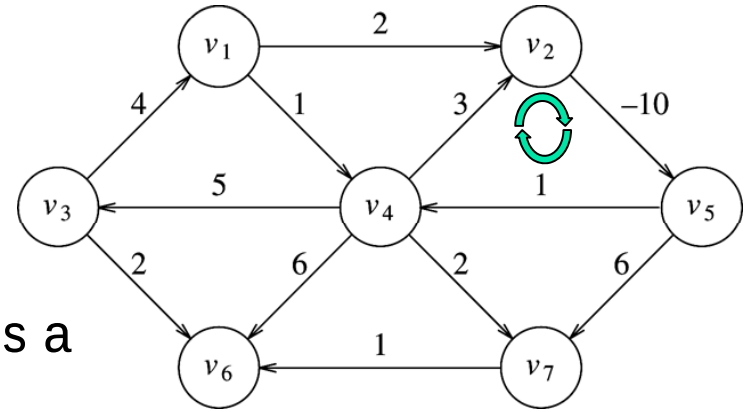
(brute-force)

Solve Single Source Shortest Path for each vertex as source

There are more efficient ways of solving this problem (e.g., Floyd-Warshall algo).

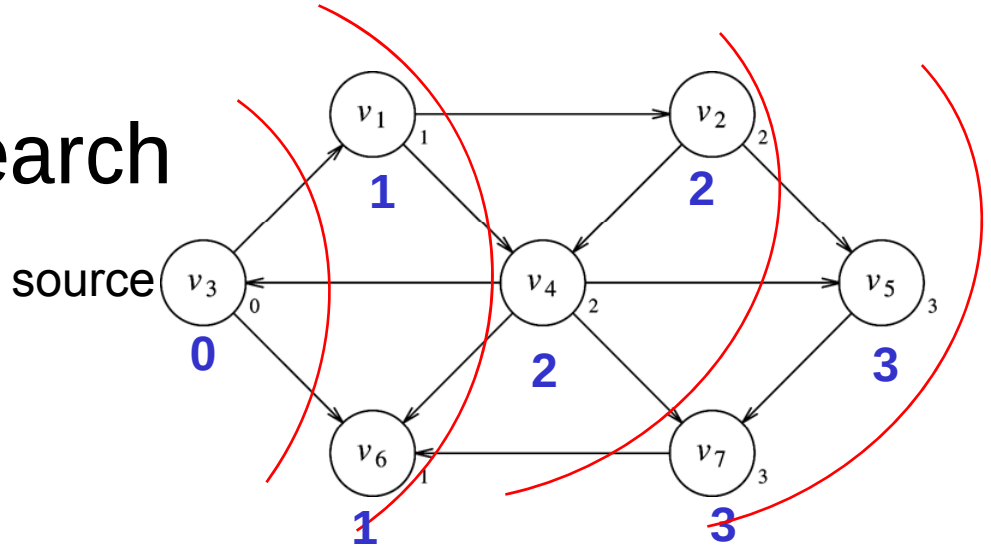
Negative Weights

- Graphs can have negative weights
- E.g., arbitrage
  - Shortest positive-weight path is a net gain
  - Path may include individual losses
- Problem: Negative weight cycles
  - Allow arbitrarily-low path costs
- Solution
  - Detect presence of negative-weight cycles



Unweighted Shortest Paths

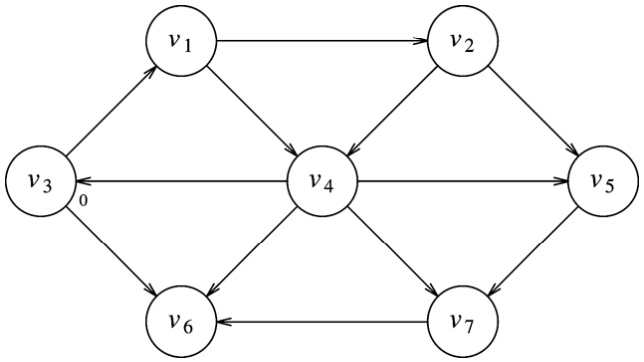
- No weights on edges
- Find shortest length paths
- Same as weighted shortest path with all weights equal
- Breadth-first search



O(|E| + |V|)

Unweighted Shortest Paths

- For each vertex, keep track of
  - Whether we have visited it (*known*)
  - Its distance from the start vertex (d_v)
  - Its predecessor vertex along the shortest path from the start vertex (p_v)



v	*known*	d_v	p_v
v_1	F	∞	0
v_2	F	∞	0
v_3	F	0	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

Unweighted Shortest Paths

```
void Graph::unweighted( Vertex s )
{
    for each Vertex v
    {
        v.dist = INFINITY;
        v.known = false;
    }

    s.dist = 0;

    for( int currDist = 0; currDist < NUM_VERTICES; currDist++ )
        for each Vertex v
            if( !v.known && v.dist == currDist )
            {
                v.known = true;
                for each Vertex w adjacent to v
                    if( w.dist == INFINITY )
                    {
                        w.dist = currDist + 1;
                        w.path = v;
                    }
            }
}
```

Solution 1: Repeatedly iterate through vertices, looking for unvisited vertices at current distance from start vertex s.

Running time: $O(|V|^2)$

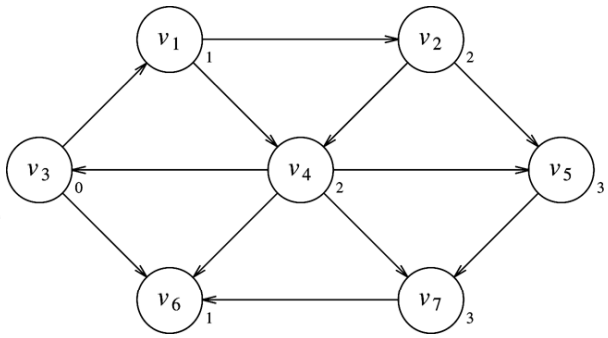
Unweighted Shortest Paths

```
void Graph::unweighted( Vertex s )
{
    Queue<Vertex> q;

    for each Vertex v
        v.dist = INFINITY;

    s.dist = 0;
    q.enqueue( s );

    while( !q.isEmpty( ) )
    {
        Vertex v = q.dequeue( );

        for each Vertex w adjacent to v
            if( w.dist == INFINITY )
            {
                w.dist = v.dist + 1;
                w.path = v;
                q.enqueue( w );
            }
    }
}
```

Solution: Ignore vertices that have already been visited by keeping only unvisited vertices (distance = ∞) on the queue.

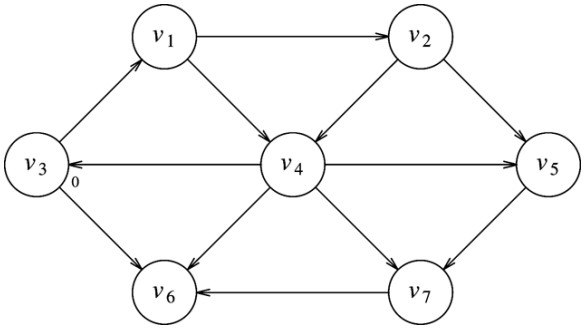
Running time: O(|E|+|V|)

source

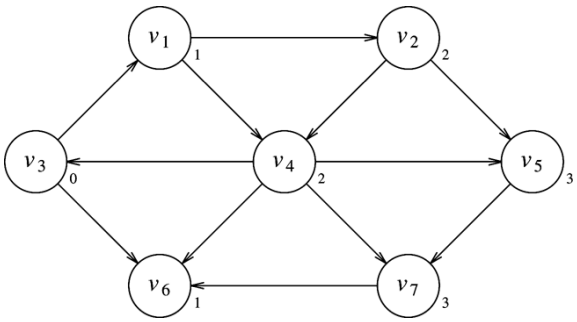


Queue: v_3 | v_1, v_6 | v_2, v_4 | v_7, v_5

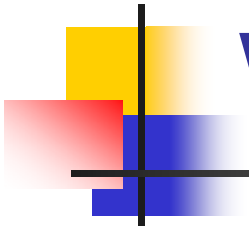
Unweighted Shortest Paths



	Initial State			v_3 Dequeued			v_1 Dequeued			v_6 Dequeued		
v	*known*	d_v	p_v	*known*	d_v	p_v	*known*	d_v	p_v	*known*	d_v	p_v
v_1	F	∞	0	F	1	v_3	T	1	v_3	T	1	v_3
v_2	F	∞	0	F	∞	0	F	2	v_1	F	2	v_1
v_3	F	0	0	T	0	0	T	0	0	T	0	0
v_4	F	∞	0	F	∞	0	F	2	v_1	F	2	v_1
v_5	F	∞	0	F	∞	0	F	∞	0	F	∞	0
v_6	F	∞	0	F	1	v_3	F	1	v_3	T	1	v_3
v_7	F	∞	0	F	∞	0	F	∞	0	F	∞	0
Q:	v_3			v_1, v_6			v_6, v_2, v_4			v_2, v_4		



	v_2 Dequeued			v_4 Dequeued			v_5 Dequeued			v_7 Dequeued		
v	known	d_v	p_v	*known*	d_v	p_v	*known*	d_v	p_v	*known*	d_v	p_v
v_1	T	1	v_3	T	1	v_3	T	1	v_3	T	1	v_3
v_2	T	2	v_1	T	2	v_1	T	2	v_1	T	2	v_1
v_3	T	0	0	T	0	0	T	0	0	T	0	0
v_4	F	2	v_1	T	2	v_1	T	2	v_1	T	2	v_1
v_5	F	3	v_2	F	3	v_2	T	3	v_2	T	3	v_2
v_6	T	1	v_3	T	1	v_3	T	1	v_3	T	1	v_3
v_7	F	∞	0	F	3	v_4	F	3	v_4	T	3	v_4
Q:	v_4, v_5			v_5, v_7			v_7			empty		



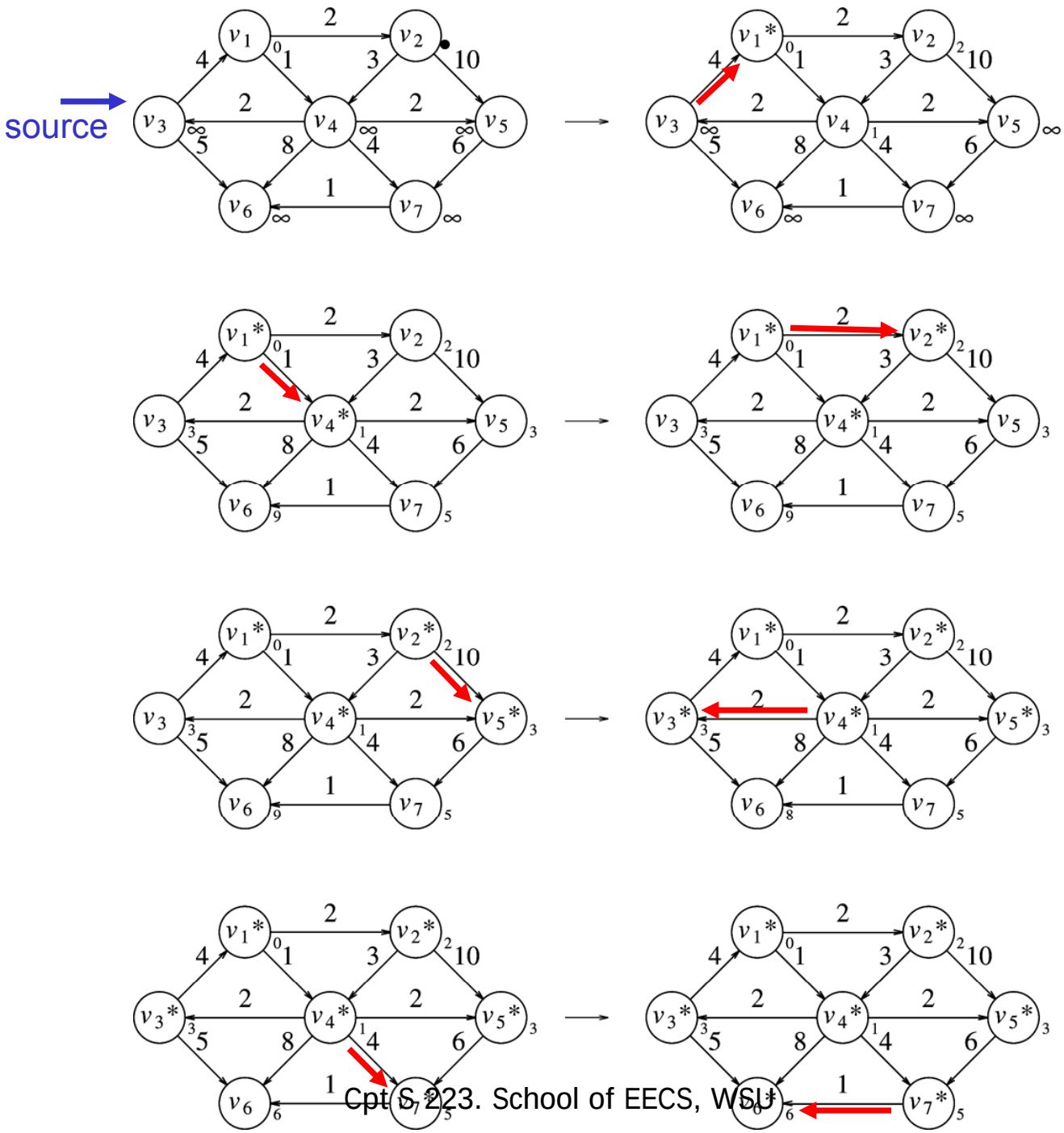
Weighted Shortest Paths

- Dijkstra's algorithm
  - GREEDY strategy:
    - Always pick the next closest vertex to the source
  - Use priority queue to store unvisited vertices by distance from s
  - After deleteMin v, update distances of remaining vertices adjacent to v using decreaseKey
  - Does not work with negative weights

Dijkstra's Algorithm

```
/**
 * PSEUDOCODE sketch of the Vertex structure.
 * In real C++, path would be of type Vertex *,
 * and many of the code fragments that we describe
 * require either a dereferencing * or use the
 * -> operator instead of the . operator.
 * Needless to say, this obscures the basic algorithmic ideas.
 */
struct Vertex
{
    List      adj;     // Adjacency list
    bool      known;
    DistType  dist;    // DistType is probably int
    Vertex    path;    // Probably Vertex *, as mentioned above
        // Other data and member functions as needed
};
```

Dijkstra



```
void Graph::dijkstra( Vertex s )
{
    for each Vertex v
    {
        v.dist = INFINITY;
        v.known = false;
    }

    s.dist = 0;

    for( ; ; )
    {
        Vertex v = smallest unknown distance vertex;
        if( v == NOT_A_VERTEX )
            break;
        v.known = true;

        for each Vertex w adjacent to v
            if( !w.known )
                if( v.dist + cvw < w.dist )
                {
                    // Update w
                    decrease( w.dist to v.dist + cvw );
                    w.path = v;
                }
    }
}
```

BuildHeap: O(|V|)

DeleteMin: O(|V| log |V|)

DecreaseKey: O(|E| log |V|)

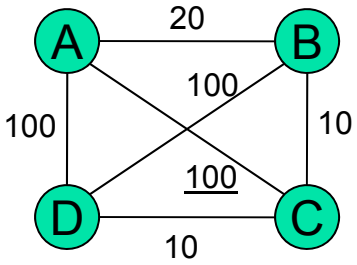
Total running time: O(|E| log |V|)

Why Dijkstra Works

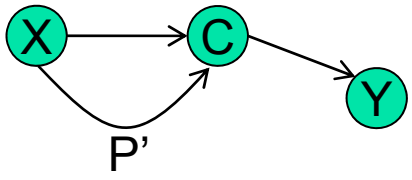
- Hypothesis

This is called the "Optimal Substructure" property

  - A least-cost path from X to Y contains least-cost paths from X to every city on the path to Y
  - E.g., if X→C1→C2→C3→Y is the least-cost path from X to Y, then
    - X→C1→C2→C3 is the least-cost path from X to C3
    - X→C1→C2 is the least-cost path from X to C2
    - X→C1 is the least-cost path from X to C1



Why Dijkstra Works



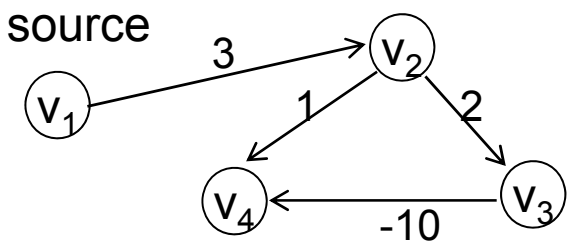
- PROOF BY CONTRADICTION:
- Assume hypothesis is false
  - I.e., Given a least-cost path P from X to Y that goes through C, there is a better path P' from X to C than the one in P
- Show a contradiction
  - But we could replace the subpath from X to C in P with this lesser-cost path P'
  - The path cost from C to Y is the same
  - Thus we now have a better path from X to Y
  - But this violates the assumption that P is the least-cost path from X to Y
- Therefore, the original hypothesis must be true

Printing Shortest Paths

```
/**
 * Print shortest path to v after dijkstra has run.
 * Assume that the path exists.
 */
void Graph::printPath( Vertex v )
{
    if( v.path != NOT_A_VERTEX )
    {
        printPath( v.path );
        cout << " to ";
    }
    cout << v;
}
```

What about graphs with negative edges?

- Will the O(|E| log|V|) Dijkstra's algorithm work as is?



deleteMin	Updates to dist
V_1	V_2.dist = 3
V_2	V_4.dist = 4, V_3.dist = 5
V_4	No change
V_3	No change and so v_4. dist will remain 4. Correct answer: v_4.dist should be updated to -5

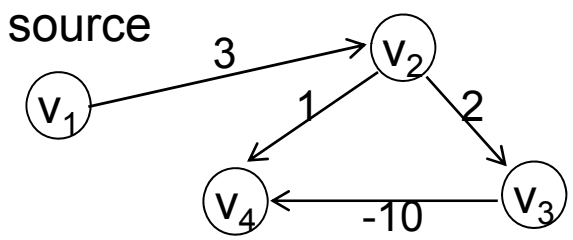
Solution:

Do not mark any vertex as "known".
Instead allow multiple updates.

Negative Edge Costs

```
void Graph::weightedNegative( Vertex s )
{
    Queue<Vertex> q;

    for each Vertex v
        v.dist = INFINITY;

    s.dist = 0;
    q.enqueue( s );

    while( !q.isEmpty( ) )
    {
        Vertex v = q.dequeue( );

        for each Vertex w adjacent to v
            if( v.dist + cvw < w.dist )
            {
                // Update w
                w.dist = v.dist + cvw;
                w.path = v;
                if( w is not already in q )
                    q.enqueue( w );
            }
    }
}
```

Running time: O(|E|·|V|)



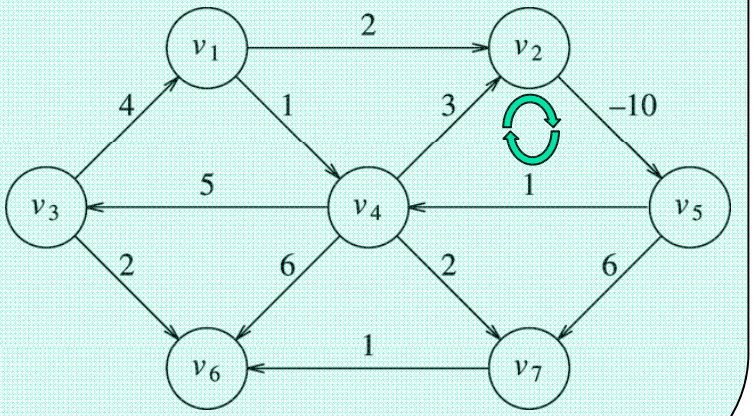
Queue	Dequeue	Updates to dist
V_1	V_1	V_2.dist = 3
V_2	V_2	V_4.dist = 4, V_3.dist = 5
V_4, V_3	V_4	No updates
V_3	V_3	V_4.dist = -5
V_4	V_4	No updates

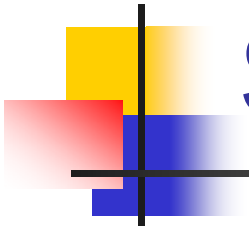
Negative Edge Costs

```
void Graph::weightedNegative( Vertex s )
{
    Queue<Vertex> q;

    for each Vertex v
        v.dist = INFINITY;

    s.dist = 0;
    q.enqueue( s );

    while( !q.isEmpty( ) )
    {
        Vertex v = q.dequeue( );

        for each Vertex w adjacent to v
            if( v.dist + cvw < w.dist )
            {
                // Update w
                w.dist = v.dist + cvw;
                w.path = v;
                if( w is not already in q )
                    q.enqueue( w );
            }
    }
}
```

Running time: O(|E|·|V|)

Negative weight cycles?





Shortest Path Problems

- Unweighted shortest-path problem: O(|E|+|V|)
- Weighted shortest-path problem
  - No negative edges: O(|E| log |V|)
  - Negative edges: O(|E|·|V|)
- Acyclic graphs: O(|E|+|V|)
- No asymptotically faster algorithm for single-source/single-destination shortest path problem

Course Evaluation Site in now Open!

http://skylight.wsu.edu/s/053eadf6-6157-44ce-92ad-cbc26bde3b53.srv